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Asymptotic behavior of solutions of a first-order impulsive neutral differential equation in Euler form

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ABSTRACT

This paper is concerned with an impulsive neutral differential equation of Euler form with unbounded delays

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}[x(t) - C(t)x(\alpha t)] + \dfrac{P(t)}{t}x(\beta t) = 0, \quad t \geq t_0 > 0,\ t \neq t_k, \\ x(t_k) = b_k x(t_k^-) + (1 - b_k)\displaystyle\int_{\beta t_k}^{t_k} \dfrac{P(s/\beta)}{s}x(s)\mathrm{d}s, \quad k = 1, 2, \ldots. \end{cases} \tag{$*$}$$

Sufficient conditions are obtained for every solution of ($*$) to tend to a constant as $t \to \infty$.

1. Introduction

Impulsive differential equations are now recognized as an excellent source of models to simulate processes and phenomena observed in control theory, physics, chemistry, population dynamics, industrial robotics, economics, etc. [1,2]. In recent years, there has been increasing interest in the oscillation and stability theory of impulsive delay differential equations and many results have been obtained (see [3–6] and the references cited therein). In particular, stability of some impulsive neutral differential equations with constant delays has also been studied by several authors [7,8]. However, to the best of our knowledge, there is very little in the way of results for the asymptotic behavior of solutions of impulsive neutral differential equations with variable delays though there are many results on the qualitative properties of delay differential equations with variable delays [9,10].

In this paper, we consider the asymptotic behavior of solutions of the following neutral differential equation in Euler form with impulsive perturbations

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t) - C(t)x(\alpha t)] + \frac{P(t)}{t}x(\beta t) = 0, \quad t \geq t_0 > 0,\ t \neq t_k, \tag{1.1}$$

$$x(t_k) = b_k x(t_k^-) + (1 - b_k)\int_{\beta t_k}^{t_k} \frac{P(s/\beta)}{t}x(s)\mathrm{d}s, \quad k = 1, 2, \ldots, \tag{1.2}$$

where $0 < \alpha, \beta < 1$, $C(t), P(t) \in PC([t_0, \infty), R)$ and $P(t) \geq 0$, $0 < t_0 < t_k < t_{k+1}$, with $\lim_{t\to\infty} t_k = \infty$, and b_k, $k = 1, 2, \ldots$, are constants. $PC([t_0, \infty), R)$ denotes the set of all functions $g : [t_0, \infty) \to R$ such that g is continuous for $t_k \leq t < t_{k+1}$ and $\lim_{t\to t_k^-} g(t) = g(t_k^-)$.

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In system (1.1) and (1.2) the impulsive term is also delayed, that is, it contains an integral term. When all $b_k = 1$, $k = 1, 2, \ldots$, the system (1.1) and (1.2) reduces to the first-order neutral differential equation of Euler form

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t) - C(t)x(\alpha t)] + \frac{P(t)}{t}x(\beta t) = 0, \quad t \geq t_0 > 0. \tag{1.3}$$

The oscillatory behavior of solutions of (1.3) and its corresponding differential equation with certain impulsive perturbations was investigated in [11,12], respectively. However, the asymptotic behavior of solutions of such equations is not still studied. Thus, there is strong interest in investigating such problems.

The main purpose of this paper is to investigate the asymptotic behavior of solutions of the system (1.1) and (1.2). As a consequence, some sufficient conditions are obtained for the asymptotic stability of solutions of (1.3).

With the system (1.1) and (1.2), one associates an initial condition of the form

$$x_{t_0} = \phi(\eta), \quad \eta \in [\rho, 1], \tag{1.4}$$

where $\rho = \min\{\alpha, \beta\}$, $x_{t_0} = x(\eta t_0)$ for $\rho \leq \eta \leq 1$ and $\phi \in PC([\rho, 1], R) = \{\phi : [\rho, 1] \to R | \phi$ is continuous everywhere except at a finite number of points $\bar{\eta}$, and $\phi(\bar{\eta}^-)$ and $\phi(\bar{\eta}^+) = \lim_{\eta \to \bar{\eta}^+} \phi(\eta)$ exist with $\phi(\bar{\eta}^+) = \phi(\bar{\eta})\}$.

A function $x(t)$ is said to be a solution of (1.1) and (1.2) satisfying the initial value condition (1.4) if

(i) $x(t) = \phi(t/t_0)$ for $\rho t_0 \leq t \leq t_0$, $x(t)$ is continuous for $t \geq t_0$ and $t \neq t_k$, $k = 1, 2, \ldots$;
(ii) $x(t) - C(t)x(\alpha t)$ is continuously differentiable for $t > t_0$, $t \neq t_k$, $k = 1, 2, \ldots$, and satisfies (1.1);
(iii) $x(t_k^+)$ and $x(t_k^-)$ exist with $x(t_k^+) = x(t_k)$ and satisfy (1.2).

Using the method of steps as in the case without impulses, one can show the global existence and uniqueness of the solution of the initial problem (1.1), (1.2) and (1.4).

A solution of (1.1) and (1.2) is said to be nonoscillatory if it is eventually positive or eventually negative. Otherwise, the solution is said to be oscillatory.

2. Main results

Theorem 2.1. *Assume that the following conditions hold:*

$$\alpha t_k \text{ is not an impulsive point}, \ 0 < b_k \leq 1 (k = 1, 2, \ldots) \text{ and } \sum_{k=1}^{\infty}(1 - b_k) < \infty; \tag{2.1}$$

$$C(t_k) = b_k C(t_k^-); \tag{2.2}$$

$$\lim_{t\to\infty} |C(t)| = \mu < 1; \tag{2.3}$$

$$\limsup_{t\to\infty} \left[\mu\left(1 + \frac{P(t/\alpha\beta)}{P(t/\beta)}\right) + \int_{\beta t}^{t/\beta} \frac{P(s/\beta)}{s}\mathrm{d}s\right] < 2. \tag{2.4}$$

Then every solution of (1.1) *and* (1.2) *tends to a constant as* $t \to \infty$.

Proof. Let $x(t)$ be any solution of (1.1) and (1.2). We shall prove that $\lim_{t\to\infty} x(t)$ exists and is finite. For this purpose, we rewrite the system (1.1) and (1.2) in the form

$$\left[x(t) - C(t)x(\alpha t) - \int_{\beta t}^{t} \frac{P(s/\beta)}{s}x(s)\mathrm{d}s\right]' + \frac{P(t/\beta)}{t}x(t) = 0, \quad t \geq t_0, \ t \neq t_k, \tag{2.5}$$

$$x(t_k) = b_k x(t_k^-) + (1 - b_k)\int_{\beta t_k}^{t_k} \frac{P(s/\beta)}{s}x(s)\mathrm{d}s, \quad k = 1, 2, \ldots. \tag{2.6}$$

From (2.3) and (2.4), we can select $\lambda > 0$ and $\delta > 0$ sufficiently small and $T > t_0$ sufficiently large such that $\mu + \lambda < 1$,

$$\left[(\mu + \lambda)\left(1 + \frac{P(t/\alpha\beta)}{P(t/\beta)}\right) + \int_{\beta t}^{t/\beta} \frac{P(s/\beta)}{s}\mathrm{d}s\right] < 2 - \delta, \quad \text{for } t \geq T, \tag{2.7}$$

and

$$|C(t)| \leq \mu + \lambda, \quad \text{for } t \geq T. \tag{2.8}$$

In what follows, for the sake of convenience, when we write a functional inequality without specifying its domain of validity, we mean that it holds for all sufficiently large t. Now we introduce two functionals:

$$V_1(t) = \left[x(t) - C(t)x(\alpha t) - \int_{\beta t}^{t} \frac{P(s/\beta)}{s} x(s)\mathrm{d}s\right]^2,$$
$$V_2(t) = \int_{\beta t}^{t} \frac{P(s/\beta^2)}{s} \int_{s}^{t} \frac{P(u/\beta)}{u} x^2(u)\mathrm{d}u\mathrm{d}s + (\mu+\lambda)\int_{\alpha t}^{t} \frac{P(s/\alpha\beta)}{s} x^2(s)\mathrm{d}s.$$

At $t \neq t_k$, calculating $\frac{\mathrm{d}V_1}{\mathrm{d}t}$ and $\frac{\mathrm{d}V_2}{\mathrm{d}t}$ along the solution of (1.1), we have

$$\begin{aligned}\frac{\mathrm{d}V_1}{\mathrm{d}t} &= -2\left[x(t) - C(t)x(\alpha t) - \int_{\beta t}^{t} \frac{P(s/\beta)}{s} x(s)\mathrm{d}s\right] \frac{P(t/\beta)}{t} x(t) \\ &= -\frac{P(t/\beta)}{t}\left[2x^2(t) - 2C(t)x(t)x(\alpha t) - \int_{\beta t}^{t} \frac{P(s/\beta)}{s} 2x(s)x(t)\mathrm{d}s\right] \\ &\le -\frac{P(t/\beta)}{t}\left[2x^2(t) - |C(t)|x^2(\alpha t) - |C(t)|x^2(t) - x^2(t)\int_{\beta t}^{t} \frac{P(s/\beta)}{s}\mathrm{d}s - \int_{\beta t}^{t} \frac{P(s/\beta)}{s} x^2(s)\mathrm{d}s\right],\end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}V_2}{\mathrm{d}t} &= -\frac{P(t/\beta)}{t}\int_{\beta t}^{t} \frac{P(u/\beta)}{u} x^2(u)\mathrm{d}u + \frac{P(t/\beta)}{t} x^2(t)\int_{\beta t}^{t} \frac{P(s/\beta^2)}{s}\mathrm{d}s \\ &\quad + (\mu+\lambda)\frac{P(t/\alpha\beta)}{t} x^2(t) - (\mu+\lambda)\frac{P(t/\beta)}{t} x^2(\alpha t).\end{aligned}$$

Let $V(t) = V_1(t) + V_2(t)$. From the above two inequalities and (2.7) and (2.8), it follows that

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} &= \frac{\mathrm{d}V_1}{\mathrm{d}t} + \frac{\mathrm{d}V_2}{\mathrm{d}t} \\ &\le -\frac{P(t/\beta)}{t}\left[2x^2(t) - |C(t)|x^2(t) - x^2(t)\int_{\beta t}^{t} \frac{P(s/\beta)}{s}\mathrm{d}s \right. \\ &\quad \left. - x^2(t)\int_{\beta t}^{t} \frac{P(s/\beta^2)}{s}\mathrm{d}s\right] + (\mu+\lambda)\frac{P(t/\alpha\beta)}{t} x^2(t) \\ &= -\frac{P(t/\beta)}{t} x^2(t)\left[2 - |C(t)| - \int_{\beta t}^{t/\beta} \frac{P(s/\beta)}{s}\mathrm{d}s - (\mu+\lambda)\frac{P(t/\alpha\beta)}{P(t/\beta)}\right] \\ &\le -\frac{P(t/\beta)}{t} x^2(t)\left[2 - \int_{\beta t}^{t/\beta} \frac{P(s/\beta)}{s}\mathrm{d}s - (\mu+\lambda)\left(1 + \frac{P(t/\alpha\beta)}{P(t/\beta)}\right)\right] \\ &\le -\delta\frac{P(t/\beta)}{t} x^2(t).\end{aligned} \tag{2.9}$$

At $t = t_k$, from (1.2) and (2.1), it follows that

$$\begin{aligned}V(t_k) &= \left[x(t_k) - C(t_k)x(\alpha t_k) - \int_{\beta t_k}^{t_k} \frac{P(s/\beta)}{s} x(s)\mathrm{d}s\right]^2 \\ &\quad + \int_{\beta t_k}^{t_k} \frac{P(s/\beta^2)}{s}\int_{s}^{t_k} \frac{P(u/\beta)}{u} x^2(u)\mathrm{d}u\mathrm{d}s + (\mu+\lambda)\int_{\alpha t_k}^{t_k} \frac{P(s/\alpha\beta)}{s} x^2(s)\mathrm{d}s \\ &= b_k^2\left[x(t_k^-) - C(t_k^-)x(\alpha t_k) - \int_{\beta t_k}^{t_k} \frac{P(s/\beta)}{s} x(s)\mathrm{d}s\right]^2 \\ &\quad + \int_{\beta t_k}^{t_k} \frac{P(s/\beta^2)}{s}\int_{s}^{t_k} \frac{P(u/\beta)}{u} x^2(u)\mathrm{d}u\mathrm{d}s + (\mu+\lambda)\int_{\alpha t_k}^{t_k} \frac{P(s/\alpha\beta)}{s} x^2(s)\mathrm{d}s \\ &\le V(t_k^-).\end{aligned} \tag{2.10}$$

This and (2.9) show that $V(t)$ is decreasing. In view of the fact that $V(t) \ge 0$, $\lim_{t\to\infty} V(t) = \gamma$ exists and $\gamma \ge 0$. By using (2.9) and (2.10) again, we can easily get

$$\int_{T}^{\infty} \frac{P(t/\beta)}{t} x^2(t)\mathrm{d}t \le \frac{V(T)}{\delta}.$$

This implies that

$$\frac{P(t/\beta)}{t}x^2(t) \in L^1(t_0, \infty),$$

and hence for any $0 < \eta < 1$ we have $\lim_{t\to\infty} \int_{\eta t}^{t} \frac{P(s/\beta)}{s}x^2(s)\mathrm{d}s = 0$. Thus, it follows from (2.4) that

$$\begin{aligned}
\int_{\beta t}^{t} \frac{P(s/\beta^2)}{s} \int_{s}^{t} \frac{P(u/\beta)}{u}x^2(u)\mathrm{d}u\mathrm{d}s &\le \int_{\beta t}^{t} \frac{P(s/\beta^2)}{s}\mathrm{d}s \int_{\beta t}^{t} \frac{P(u/\beta)}{u}x^2(u)\mathrm{d}u \\
&= \int_{t}^{t/\beta} \frac{P(s/\beta)}{s}\mathrm{d}s \int_{\beta t}^{t} \frac{P(u/\beta)}{u}x^2(u)\mathrm{d}u \\
&\le \int_{\beta t}^{t/\beta} \frac{P(s/\beta)}{s}\mathrm{d}s \int_{\beta t}^{t} \frac{P(u/\beta)}{u}x^2(u)\mathrm{d}u \\
&\le 2\int_{\beta t}^{t} \frac{P(u/\beta)}{u}x^2(u)\mathrm{d}u \to 0, \quad \text{as } t \to \infty,
\end{aligned}$$

and

$$\begin{aligned}
(\mu+\lambda)\int_{\alpha t}^{t} \frac{P(s/\alpha\beta)}{s}x^2(s)\mathrm{d}s &= (\mu+\lambda)\int_{\alpha t}^{t} \frac{P(s/\alpha\beta)}{P(s/\beta)} \cdot \frac{P(s/\beta)}{s}x^2(s)\mathrm{d}s \\
&\le 2\int_{\alpha t}^{t} \frac{P(s/\beta)}{s}x^2(s)\mathrm{d}s \to 0, \quad \text{as } t \to \infty.
\end{aligned}$$

It follows that $\lim_{t\to\infty} V_2(t) = 0$. Thus, $\lim_{t\to\infty} V_1(t) = \lim_{t\to\infty} V(t) = \gamma$, that is,

$$\lim_{t\to\infty}\left[x(t) - C(t)x(\alpha t) - \int_{\beta t}^{t} \frac{P(s/\beta)}{s}x(s)\mathrm{d}s\right]^2 = \gamma. \tag{2.11}$$

Next, we shall prove that the limit

$$\lim_{t\to\infty}\left[x(t) - C(t)x(\alpha t) - \int_{\beta t}^{t} \frac{P(s/\beta)}{s}x(s)\mathrm{d}s\right]$$

exists and is finite. Set $y(t) = x(t) - C(t)x(\alpha t) - \int_{\beta t}^{t} \frac{P(s/\beta)}{s}x(s)\mathrm{d}s$, from (1.2) and (2.1), we then have

$$\begin{aligned}
y(t_k) &= x(t_k) - C(t_k)x(\alpha t_k) - \int_{\beta t_k}^{t_k} \frac{P(s/\beta)}{s}x(s)\mathrm{d}s \\
&= b_k\left[x(t_k^-) - C(t_k^-)x(\alpha t_k) - \int_{\beta t_k}^{t_k} \frac{P(s/\beta)}{s}x(s)\mathrm{d}s\right] \\
&= b_k y(t_k^-).
\end{aligned} \tag{2.12}$$

From (2.11), it follows that

$$\lim_{t\to\infty} y^2(t) = \gamma. \tag{2.13}$$

Moreover, in view of (2.5) and (2.12), system (2.5) and (2.6) can be rewritten as

$$\begin{cases} y'(t) + \dfrac{P(t/\beta)}{t}x(t) = 0, \quad t \ge t_0 > 0,\ t \ne t_k, \\ y(t_k) = b_k y(t_k^-), \quad k = 1, 2, \ldots. \end{cases} \tag{2.14}$$

If $\gamma = 0$, then $\lim_{t\to\infty} y(t) = 0$. If $\gamma > 0$, then there exists a sufficiently large T_1 such that $y(t) \ne 0$ for $t \ge T_1$. Otherwise, there is a sequence $\{\tau_k\}$ with $\lim_{k\to\infty} \tau_k = \infty$ such that $y(\tau_k) = 0$, and so $y^2(\tau_k) \to 0$ as $k \to \infty$. This contradicts $\gamma > 0$. Therefore, for any $t_k > T_1$, $t \in [t_k, t_{k+1})$, we have $y(t) > 0$ or $y(t) < 0$ because $y(t)$ is continuous on $[t_k, t_{k+1})$. Without loss of generality, we assume that $y(t) > 0$ on $[t_k, t_{k+1})$. It follows that $y(t_{k+1}) = b_k y(t_{k+1}^-) > 0$, and thus $y(t) > 0$ on $[t_{k+1}, t_{k+2})$. By induction, we can conclude that $y(t) > 0$ on $[t_k, \infty)$. This and (2.11) imply that

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty}\left(x(t) - C(t)x(\alpha t) - \int_{\beta t}^{t} \frac{P(s/\beta)}{s}x(s)\mathrm{d}s\right) = \nu, \tag{2.15}$$

where $\nu = \sqrt{\gamma}$ and is finite. In view of (2.14), we have

$$\int_{\beta t}^{t} \frac{P(s/\beta)}{s} x(s)\mathrm{d}s = y(\beta t) - y(t) - \sum_{\beta t < t_k < t} (y(t_k) - y(t_k^-)) = y(\beta t) - y(t) - \sum_{\beta t < t_k < t} (1 - b_k) y(t_k^-).$$

Letting $t \to \infty$ and noticing that $\sum_{k=1}^{\infty}(1 - b_k) < \infty$, we have

$$\lim_{t\to\infty} \int_{\beta t}^{t} \frac{P(s/\beta)}{s} x(s)\mathrm{d}s = 0. \tag{2.16}$$

It follows from (2.15) and (2.16) that

$$\lim_{t\to\infty} (x(t) - C(t)x(\alpha t)) = \nu. \tag{2.17}$$

Finally, we shall prove that

$$\lim_{t\to\infty} x(t) \text{ exists and is finite.} \tag{2.18}$$

To this end, we need to show that $|x(t)|$ is bounded. As a matter of fact, if $|x(t)|$ is unbounded, then there exists a sequence $\{s_n\}$ such that $s_n \to \infty$, $|x(s_n^-)| \to \infty$ as $n \to \infty$, and

$$|x(s_n^-)| = \sup_{t_0 \le t \le s_n} |x(s_n)|,$$

where, if s_n is not an impulsive point, then $x(s_n^-) = x(s_n)$. Thus, we have

$$|x(s_n^-) - C(s_n^-)x(\alpha s_n)| \ge |x(s_n^-)| - |C(s_n^-)||x(\alpha s_n)| \ge |x(s_n^-)|(1 - \mu - \lambda) \to \infty, \quad \text{as } n \to \infty,$$

which contradicts (2.17) and so $|x(t)|$ is bounded.

If $\mu = 0$, clearly $\lim_{t\to\infty} x(t) = \nu$, which shows that (2.18) holds.

If $0 < \mu < 1$, one can see that $C(t)$ is eventually positive or negative. Otherwise, there is a sequence $\{\tau_k\}$ with $\lim_{k\to\infty} \tau_k = \infty$ such that $C(\tau_k) = 0$, and so $C(\tau_k) \to 0$ as $k \to \infty$. It is a contradiction with $\mu > 0$.

By condition (2.3), one can find a sufficiently large T_2 such that for $t > T_2$, $|C(t)| < 1$. Set

$$\xi = \liminf_{t\to\infty} x(t), \qquad \varrho = \limsup_{t\to\infty} x(t).$$

Then we can select two sequences $\{u_n\}$ and $\{v_n\}$ such that $u_n \to \infty$, $v_n \to \infty$ as $n \to \infty$, and

$$\lim_{n\to\infty} x(u_n) = \xi, \qquad \lim_{n\to\infty} x(v_n) = \varrho.$$

For $t > T_2$, we consider the following two possible cases. □

Case 1. When $-1 < C(t) < 0$ for $t > T_2$, we have

$$\nu = \lim_{n\to\infty} [x(u_n) - C(u_n)x(\alpha u_n)] \le \xi + \mu\varrho,$$

and

$$\nu = \lim_{n\to\infty} [x(v_n) - C(v_n)x(\alpha v_n)] \ge \varrho + \mu\xi.$$

Thus, we obtain

$$\xi + \mu\varrho \ge \varrho + \mu\xi,$$

that is,

$$(1 - \mu)\xi \ge (1 - \mu)\varrho.$$

Since $0 < \mu < 1$ and $\varrho \ge \xi$, it follows that $\varrho = \xi$. It follows from (2.17) that

$$\varrho = \xi = \frac{\nu}{1 + \mu},$$

which shows that (2.18) holds.

Case 2. When $0 < C(t) < 1$ for $t > T_2$, we have

$$\xi = \lim_{n\to\infty} x(u_n) = \lim_{n\to\infty} [(x(u_n) - C(u_n)x(\alpha u_n)) + C(u_n)x(\alpha u_n)] \geq \nu + \mu\xi.$$

It follows that

$$\xi \geq \frac{\nu}{1-\mu}.$$

Similarly,

$$\varrho = \lim_{n\to\infty} x(v_n) = \lim_{n\to\infty} [(x(v_n) - C(v_n)x(\alpha v_n)) + C(v_n)x(\alpha v_n)] \leq \nu + \mu\varrho,$$

which implies $\varrho \leq \frac{\nu}{1-\mu}$. Thus, $\xi = \varrho = \frac{\nu}{1-\mu}$ and so (2.18) holds.

Summarizing the above discussion, we conclude that (2.18) holds and so the proof is completed.

By Theorem 2.1, we have the following asymptotic behavior result immediately.

Theorem 2.2. *The conditions of Theorem 2.1 imply that every oscillatory solution of (1.1) and (1.2) tends to zero as $t \to \infty$.*

In Theorem 2.1, taking $b_k \equiv 1$, $k = 1, 2, \ldots$, we obtain the following

Corollary 2.3. *Assume that (2.3) and (2.4) hold. Then every solution of (1.3) tends to a constant as $t \to \infty$.*

Theorem 2.4. *The conditions of Theorem 2.1 together with*

$$\int_{t_0}^{\infty} \frac{P(s/\beta)}{s} \mathrm{d}s = \infty \tag{2.19}$$

imply that every solution of (1.1) and (1.2) tends to zero as $t \to \infty$.

Proof. By Theorem 2.2, we only have to prove that every nonoscillatory solution of (1.1) and (1.2) tends to zero as $t \to \infty$. Without loss of generality, let $x(t)$ be an eventually positive solution of (1.1) and (1.2), we shall prove that $\lim_{t\to\infty} x(t) = 0$. As in the proof of Theorem 2.1, we can rewrite (1.1) and (1.2) in the form (2.14). Integrating from t_0 to t both sides of (2.14) yields

$$\int_{t_0}^{t} \frac{P(s/\beta)}{s} x(s)\mathrm{d}s = y(t_0) - y(t) - \sum_{t_0<t_k<t} (1-b_k)y(t_k^-).$$

Using (2.15) and $\sum_{k=1}^{\infty}(1-b_k) < \infty$, we obtain

$$\int_{t_0}^{\infty} \frac{P(s/\beta)}{s} x(s)\mathrm{d}s < \infty.$$

This, together with (2.19), implies that $\liminf_{t\to\infty} x(t) = 0$. By Theorem 2.1, $\lim_{t\to\infty} x(t) = 0$ and so the proof is completed.

The following corollary follows from Corollary 2.3 and Theorem 2.4. □

Corollary 2.5. *Assume that (2.3), (2.4) and (2.19) hold. Then every solution of (1.3) tends to zero as $t \to \infty$.*

3. Examples

In this section, we give two examples to illustrate the usefulness of our main results.

Example 3.1. Consider the neutral differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t) - \frac{1}{4}x(t/2)\right] + \frac{1}{8t}x(t/4) = 0, \quad t \geq 1. \tag{3.1}$$

One can easily see that the conditions (2.3), (2.4) and (2.19) hold. By Corollary 2.5, every solution of (3.1) tends to zero as $t \to \infty$. Indeed, $x(t) = 1/t (t \geq 1)$ is such a solution.

Example 3.2. Consider the impulsive neutral differential equation

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}[x(t) - C(t)x(t/e)] + \dfrac{1}{4t(\ln t - 1)}x(t/e) = 0, & t \geq t_0 = 3, \ t \neq k, \\ x(k) = \dfrac{k^2-1}{k^2}x(k^-) + \dfrac{1}{k^2}\displaystyle\int_{k/e}^{k} \frac{1}{4s\ln s}x(s)\mathrm{d}s, & k = 4, 5, \ldots, \end{cases} \tag{3.2}$$

where $P(t) = \frac{1}{4(\ln t - 1)}$, $b_k = \frac{k^2-1}{k^2}$, and $C(t) = \frac{k}{4(k-1)^2}[t]$, $t \in [k-1, k)$, $k = 4, 5, \ldots$.

One can easily find that

$$\lim_{t\to\infty} |C(t)| = \frac{1}{4} < 1, \qquad C(k) = \frac{k^2-1}{k^2} C(k^-), \qquad \int_{t_0}^{\infty} \frac{P(s/\beta)}{s} \mathrm{d}s = \int_3^{\infty} \frac{1}{4s\ln s} \mathrm{d}s = \infty,$$

and

$$\begin{aligned}
\limsup_{t\to\infty} \left[\mu\left(1 + \frac{P(t/\alpha\beta)}{P(t/\beta)}\right) + \int_{\beta t}^{t/\beta} \frac{P(s/\beta)}{s} \mathrm{d}s\right] &= \limsup_{t\to\infty} \left[\frac{1}{4}\left(1 + \frac{\ln t}{1+\ln t}\right) + \int_{t/e}^{et} \frac{1}{4s\ln s} \mathrm{d}s\right] \\
&= \limsup_{t\to\infty} \frac{1}{4}\left(1 + \frac{\ln t}{1+\ln t} + \ln\left(\frac{\ln t+1}{\ln t-1}\right)\right) \\
&= \frac{1}{2} < 2.
\end{aligned}$$

By Theorem 2.4, every solution of (3.2) tends to zero as $t \to \infty$.

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